The role of correlation in diffusion control ranking games

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based on joint work with
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A diffusion control problem

- state dynamics:
$$dX_t^\alpha = \alpha_t dW_t$$
- controls α with values in $[\sigma_1, \sigma_2]$, where $0 < \sigma_1 < \sigma_2$
- target:
$$P(X_T^\alpha > 0) \longrightarrow max!$$

Questions:

1. Optimal control = ?
2. maximal probability if $X_0 = 0$?

Why interesting ?

Problem faced by

- managers
- research and development teams
- sports teams

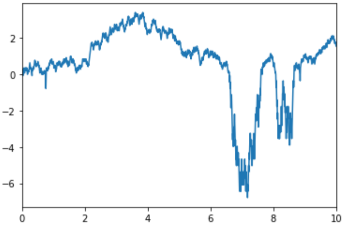
Solution of the control problem

Theorem

The control with feedback function

$$\sigma^*(x) = \begin{cases} \sigma_1, & \text{if } x \geq 0, \\ \sigma_2, & \text{if } x < 0, \end{cases}$$

is optimal.



Moreover,

$$\max_{\alpha} P(X_T^{\alpha} > 0 | X_0^{\alpha} = 0) = \frac{\sigma_2}{\sigma_1 + \sigma_2}.$$

Lessons from the control problem

- all-or-nothing payoff incentivizes agents to take risk if things are going badly
- confirms a known rule from sports: take risk if behind, play safe if ahead
- diffusion control allows to change skewness and quantiles (but not the mean)

What if the payoff depends on the performance of other agents?

- management: bonus if the own company performs better than other companies
- research: the best results will be published or put into production
- sports games: a team wins if it has more points than the other team
- elections: a candidate is elected if she has more votes than another candidate

A 2-player game

- State of player 1:

$$dX_t = \alpha(X_t, Y_t)dW_t^1, \quad X_0 = 0$$

- State of player 2:

$$dY_t = \beta(X_t, Y_t)dW_t^2, \quad Y_0 = 0$$

- $\alpha, \beta : \mathbb{R}^2 \to [\sigma_1, \sigma_2]$ **'strict controls'**
- W^1 and W^2 are BM with constant correlation

$$\rho = \mathrm{Corr}(W_t^1, W_t^2).$$

2-player game cont'd

$$\text{reward of player 1} = \begin{cases} 1, & \text{if } X_T > Y_T, \\ 0, & \text{else.} \end{cases}$$

$$\text{reward of player 2} = \begin{cases} 1, & \text{if } Y_T > X_T, \\ 0, & \text{else.} \end{cases}$$

Comments:

- Zero-sum payoff
- Players can observe the opponent's state

Which volatility controls will the players choose?

We consider first the special cases

1. $\rho = 0$
2. $\rho = 1$

Case: $\rho = 0$

In this case $D_t := X_t - Y_t$ satisfies

$$dD_t = (\alpha_t^2 + \beta_t^2)^{1/2} d\tilde{W}_t$$

Target of player 1: $P(D_T > 0) \longrightarrow \max!$

Irrespective of β_t:

- $\alpha_t = \sigma_2$ maximizes the diffusion rate
- $\alpha_t = \sigma_1$ minimizes the diffusion rate

Case: $\rho = 0$

Theorem

Let

$$\alpha^*(x, y) = \begin{cases} \sigma_1, & \text{if } x \geq y, \\ \sigma_2, & \text{if } x < y, \end{cases}$$

and

$$\beta^*(x, y) = \alpha^*(y, x).$$

Then (α^*, β^*) *is a Nash equilibrium in strict controls.*

Case: $\rho = 1$

In this case $D_t := X_t - Y_t$ satisfies

$$dD_t = (\alpha_t - \beta_t) dW_t^1$$

- If ahead, player 1 wants to choose $\alpha_t = \beta_t$.
- If behind, player 1 wants to choose α_t as far away from β_t as possible.

$\Longrightarrow$ There is no equilibrium in strict controls

Questions

1. Up to which correlation threshold does there exist an equilibrium in strict controls?
2. Can we define mixed strategies so that an equilibrium always exists?

The correlation threshold

Theorem

The game has a value in strict controls ***if and only if***

$$\rho \leq \sqrt{\frac{\sigma_1 + \sigma_2}{2\sigma_2}}. \qquad (1)$$

In this case the value function is given by

$$V_{strict}(t, x, y) = \Phi\left(\frac{x - y}{c(\rho)\sqrt{T - t}}\right), \qquad (t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R},$$

and a saddle point is given by

$$\alpha^*(x, y) = \begin{cases} \sigma_2, & \text{if } x \leq y, \\ \sigma_1 \vee \rho\sigma_2, & \text{if } x > y, \end{cases}$$

$$\beta^*(x, y) = \begin{cases} \sigma_1 \vee \rho\sigma_2, & \text{if } x \leq y, \\ \sigma_2, & \text{if } x > y. \end{cases}$$

The correlation threshold cont'd

Correlation threshold:

$$\rho \leq \sqrt{\frac{1}{2}\left(1+\frac{\sigma_1}{\sigma_2}\right)}. \tag{2}$$

The closer σ_1 and σ_2, the larger the threshold. **Why?**

- the player ahead 'mimics' with $\rho\sigma_2$
- for the player behind: σ_2 is only optimal if the alternative σ_1 is not too far below $\rho\sigma_2$.

What is the right notion of a mixed strategy in differential games?

1st attempt: randomize continuously

Problem: If $(\alpha_t)_{t\in[0,1]}$ is iid, then $(\omega, t) \mapsto \alpha_t(\omega)$ is not measurable!

2nd attempt: discretize and take limits

$$\alpha^n_t = \xi_k \qquad \text{for } t \in \left[\frac{k}{n}T, \frac{k+1}{n}\right)$$

where (ξ_k) is iid with $\sim \mu$.

Question: Where does α^n converge to?

Caution: α^n does not converge in a process space

Idea: Embed α^n into the space of **probability measures** on $[\sigma_1, \sigma_2] \times [0, T]$. The measure $\delta_{\alpha^n_t}(da)dt$ converges weakly to

$$\mu(da)dt.$$

Relaxed controls

Definition

A relaxed (Markov) control is a measurable mapping $q : [0, T] \times \mathbb{R}^2 \to \mathcal{P}([\sigma_1, \sigma_2])$.

Temptation: Define the relaxed controlled state process by

$$X_t = \int_0^t \left(\int_A a q(s, da) \right) dW_s$$

However

$$\begin{aligned} \lim_n \langle \alpha^n \cdot W, \alpha^n \cdot W \rangle_t = \lim_n \sum_{k=1}^n \frac{\xi_k^2}{n} &= \left(\int a^2 \mu(da) \right) t \qquad (LLN) \\ &\neq \left(\int a \mu(da) \right)^2 t \\ &= \langle X, X \rangle_t \end{aligned}$$

State dynamics in terms of a martingale problem

- (X_t), (Y_t) canonical processes
- $q_1, q_2 : [0, T] \times \mathbb{R}^2 \to \mathcal{P}([\sigma_1, \sigma_2])$ **'relaxed controls'**
- P^{q_1,q_2} is a feasible distribution if X and Y are martingales and

$$d\langle X, X\rangle_t = \int a^2 q_1(X_t, Y_t, da)dt$$
$$d\langle Y, Y\rangle_t = \int b^2 q_2(X_t, Y_t, db)dt$$
$$d\langle X, Y\rangle_t = \int \int \rho ab q_1(X_t, Y_t, da) q_2(X_t, Y_t, db)dt$$

Equilibria in relaxed controls

Theorem

Let $\rho > \sqrt{\frac{\sigma_1+\sigma_2}{2\sigma_2}}$. Then the game has a value in relaxed controls (given in closed form) and the tuple $(q_1^*, q_2^*) \in \mathcal{V} \times \mathcal{V}$ defined by

$$q_1^*(x,y) = \begin{cases} \frac{1}{\sigma_2-\sigma_1}\left(\left(\sigma_2 - \frac{\sigma_1+\sigma_2}{2\rho^2}\right)\delta_{\sigma_1} + \left(\frac{\sigma_1+\sigma_2}{2\rho^2} - \sigma_1\right)\delta_{\sigma_2}\right), & \text{if } x \le y, \\ \delta_{\frac{\sigma_1+\sigma_2}{2\rho}}, & \text{if } x > y, \end{cases}$$

$$q_2^*(x,y) = \begin{cases} \delta_{\frac{\sigma_1+\sigma_2}{2\rho}}, & \text{if } x \le y, \\ \frac{1}{\sigma_2-\sigma_1}\left(\left(\sigma_2 - \frac{\sigma_1+\sigma_2}{2\rho^2}\right)\delta_{\sigma_1} + \left(\frac{\sigma_1+\sigma_2}{2\rho^2} - \sigma_1\right)\delta_{\sigma_2}\right), & \text{if } x > y, \end{cases}$$

is a saddle point.

Generalizations

- Payoff of the type $g(X_T - Y_T)$, where g has only countably many discontinuities and at most exponential growth.
- More than 2 players .....

Literature

- **S. Ankirchner, N. Kazi-Tani and J. Wendt. *The role of correlation in diffusion control games*. HAL preprint (`https://hal.science/hal-03954608/`) 2023.**
- S. Ankirchner, N. Kazi-Tani, J. Wendt and Chao Zhou. *Large ranking games with diffusion control*. MOR 2023.
- S. Ankirchner, H,. Bernburg and J. Wendt. *A simple random walk game*. HAL preprint (`https://hal.science/hal-03607763/`) 2022.

Thank you!

SDE representation of relaxed controlled states

$q_1, q_2 : \mathbb{R}^2 \to \mathcal{P}([\sigma_1, \sigma_2])$ **'relaxed controls'**

Then the states solve

$$dX_t = \int_{\sigma_1}^{\sigma_2} a\, q_1(t, X_t, Y_t)(da)\, dW_t^1 + \sqrt{\mathrm{Var}(q_1(t, X_t, Y_t))}\, d\tilde{B}_t^1$$

$$dY_t = \int_{\sigma_1}^{\sigma_2} b\, q_2(t, X_t, Y_t)(db)\, dW_t^2 + \sqrt{\mathrm{Var}(q_2(t, X_t, Y_t))}\, d\tilde{B}_t^2,$$

$\tilde{B}^1, \tilde{B}^2$ new independent BMs